On the Relation between the Sum-Formulas of Hölder and Cesàro.

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I. *Introduction.*

1. If

$$u_0 + u_1 + u_2 + \dots + u_n + \dots \tag{1}$$

be a series (convergent or divergent) such that for some positive integral value of r the expression $\lim_{n=\infty} S_n^{(r)} / D_n^{(r)}$ exists and has the value l, where it is understood that

$$\left.\begin{aligned} S_n^{(r)} &= s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \dots \frac{r(r+1)\dots(r+n-1)}{n!}s_0, \\ s_n &= u_0 + u_1 + u_2 + \dots + u_n, \\ D_n^{(r)} &= \frac{(r+1)(r+2)\dots(r+n)}{n!}, \end{aligned}\right\} \tag{2}$$

then the series is said to be summable for the given value of r in Cesàro's sense, and to have the sum l.* The same series is said to be summable for a given value of r in Hölder's sense, and to have the sum l when $\lim_{n=\infty} s_n^{(r)} = l$, where $s_n^{(0)} = s_n$ and

$$s_n^{(r)} = \frac{1}{n+1}(s_0^{(r-1)} + s_1^{(r-1)} + s_2^{(r-1)} + \dots + s_n^{(r-1)}), \tag{3}$$

it being understood that s_n has the meaning already indicated.†

In the following paper it is proposed to study the relation between the sums l of (1) as given by the two preceding formulas. In this connection we shall establish the following theorem, showing that the formulas are coextensive in applicability and in their resulting sums.

THEOREM. *If the series* (1) *is summable for a specified value of r by Hölder's formula, it is summable for this value of r by Cesàro's formula, and the two sums*

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* It may be remarked that the value of the limit l is the same for all values of r for which it exists.

† Cf. Bromwich, *Infinite Series* (1908), §§ 122, 123.

are identical. Conversely, if the series (1) *is summable for a given value of r by Cesàro's formula, it is summable for this value of r by Hölder's formula, and the two sums are identical.**

II. *Reduction of the Problem.*

2. We shall first consider the existence of the above theorem when the expressions $s_n^{(0)}, s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(r)}$, instead of being defined by (3), are defined by the following analogous system of relations:

$$s_n^{(0)} = s_n; \quad s_n^{(r)} = \frac{1}{n+r}(s_0^{(r-1)} + s_1^{(r-1)} + \ldots + s_n^{(r-1)}). \tag{4}$$

To do this it is desirable for our purpose to at once determine the n coefficients $a_0(n, r)$, $a_1(n, r)$, $a_2(n, r)$, $\ldots$, $a_n(n, r)$ (each being a function of both n and r) such that for all values (positive integral) of n and r we shall have

$$S_n^{(r)} = a_0(n, r)\, s_n^{(r)} + a_1(n, r)\, s_{n-1}^{(r)} + \ldots + a_n(n, r)\, s_0^{(r)}. \tag{5}$$

Assuming for the moment the existence of such coefficients, let us suppose n to have some *fixed* value. From (2) we have

$$s_n^{(r)} = (n + r + 1)\, s_n^{(r+1)} - (n + r)\, s_{n-1}^{(r+1)},$$

so that (5) may be written in the form

$$\begin{aligned} S_n^{(r)} = (n + r + 1)a_0(n, r)\, s_n^{(r+1)} &+ (n+r)[a_1(n, r) - a_0(n, r)]\, s_{n-1}^{(r+1)} \\ &+ (n + r - 1)[a_2(n, r) - a_1(n, r)]\, s_{n-2}^{(r+1)} + \ldots \\ &+ (r + 1)[a_n(n, r) - a_{n-1}(n, r)]\, s_0^{(r+1)}. \end{aligned} \tag{6}$$

Moreover, since

$$S_n^{(r+1)} = S_n^{(r)} + S_{n-1}^{(r)} + \ldots + S_0^{(r)}$$

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*In his Dissertation entitled "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze" (1907), Knopp establishes the following result: If a series (1) is summable for a specified value of r by Hölder's formula, it is summable for the same value of r by Cesàro's formula (*l. c.*, pp. 20, 21). By use of certain established results relative to the limiting values approached by a function defined by a power series at points *on* the circle of convergence, Knopp then remarks that the two sums l must be identical. The present paper may be considered to supplement these results of Knopp in two respects: (a) it establishes the equality of the last-mentioned sums without making use of extraneous theorems on power series, thus affording a new and independent proof of Hölder's original theorem (*Math. Annalen*, Vol. XX (1882), p. 535); and (b) it considers the converse theorem.

The method which we employ was suggested by the above-mentioned investigations of Knopp. A different form of proof has recently been given by Schnee (*Math. Annalen*, Vol. LXVII (1909), pp. 110–125), the latter appearing shortly after the present paper had been finished and submitted for publication. While Schnee's proof thus has priority, it is believed that the importance of the theorem, especially in its bearing upon the theory of summable divergent series, justifies the publication of a second proof. The paper has, however, been considerably condensed owing to the above circumstance.

References to the literature of the subject may be found in the above-mentioned paper of Schnee.

we obtain by use of (6) the following relation:

$$
\begin{aligned}
S_n^{(r+1)} = {} & (n + r + 1)a_0(n, r)s_n^{(r+1)} + (n + r)\left[a_1(n, r) - a_0(n, r)\right. \\
& \left. + a_0(n - 1, r)\right] s_{n-1}^{(r+1)} + (n + r - 1)\left[a_2(n, r) - a_1(n, r)\right. \\
& \left. + a_1(n - 1, r) - a_0(n - 1, r) + a_0(n - 2, r)\right] s_{n-2}^{(r+1)} + \ldots . \\
& + (r + 1)\left[a_n(n, r) - a_{n-1}(n, r) + a_{n-1}(n - 1, r) - a_{n-2}(n - 1, r)\right. \\
& \left. + \ldots . + a_1(1, r) - a_0(1, r) + a_0(0, r)\right] s_0^{(r+1)}.
\end{aligned} \tag{7}
$$

From (5) and (7) it appears that the coefficients $a_0(n,r), a_1(n, r), \ldots, a_n(n, r)$ must satisfy the following system of equations:

$$
\begin{aligned}
a_m(n, r + 1) = (n + r - m + 1)\left[a_m(n, r) - a_{m-1}(n, r) + a_{m-1}(n - 1, r)\right. \\
- a_{m-2}(n - 1, r) + \ldots . - a_0(n - m + 1, r) \\
\left. + a_0(n - m, r)\right]; \qquad (m = 0, 1, 2, 3, \ldots, n).
\end{aligned} \tag{8}
$$

Since $S_n^{(1)} = (n + 1)s_n^{(1)}$, as results from (2) and (4), it follows that

$$a_0(n, 1) = n + 1,\ a_1(n, 1) = a_2(n, 1) = \ldots . = a_n(n, 1) = 0.$$

If we start with these values (corresponding to $r = 1$), we may now compute by means of (8) the values of $a_0(n, r),\ a_1(n, r), \ldots, a_n(n, r)$ for any given value of r.

Thus we obtain in the first place

$$a_0(n, r) = (n + r)(n + r - 1) \ldots . (n + 1); \tag{9}$$

and when $r \geqq 2 \quad (1 \leqq m \leqq n)$,*

$$
\begin{aligned}
a_m(n, r) = \sum_{g=1}^{r-1} (-1)^g \frac{(m-1)!}{(g-1)!\,(m-g)!}\sigma_g(r)(n - m + r) \\
(n - m + r - 1) \ldots . (n - m + g + 1),
\end{aligned} \tag{10}
$$

where $\sigma_1(r), \sigma_2(r), \ldots, \sigma_{r-1}(r)$ are functions of r only determined by means of the relations

$$
\left.
\begin{aligned}
\sigma_1(r) &= 1 + 2 + 3 + \ldots . + (r - 1) = \sigma_1(r - 1) + (r - 1), \\
\sigma_2(r) &= \sigma_1(2) + 2\sigma_1(3) + 3\sigma_1(4) + \ldots . + (r - 2)\sigma_1(r - 1) \\
&\qquad = \sigma_2(r - 1) + (r - 2)\sigma_1(r - 1), \\
\sigma_3(r) &= \sigma_2(3) + 2\sigma_2(4) + 3\sigma_2(5) + \ldots . + (r - 3)\sigma_2(r - 1) \\
&\qquad = \sigma_3(r - 1) + (r - 3)\sigma_2(r - 1) \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots , \\
\sigma_{r-1}(r) &= \sigma_{r-2}(r - 1) = 1.
\end{aligned}
\right\} \tag{11}
$$

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* In the condensation of the present paper already referred to, the actual demonstration by mathematical induction has been omitted.

Similarly, it is desirable to at once determine the set of coefficients $\alpha_0(n, r)$, $\alpha_1(n, r)$, ..., $\alpha_n(n, r)$ for which the following equation is satisfied for all (positive integral) values of n and r, it being understood that $s_0^{(r)}, s_1^{(r)}, \ldots, s_n^{(r)}$ are determined by (4):

$$s_n^{(r)} = \alpha_0(n, r)\, S_n^{(r)} + \alpha_1(n, r)\, S_{n-1}^{(r)} + \ldots + \alpha_n(n, r)\, S_0^{(r)}. \tag{12}$$

Since $S_n^{(r)} = S_n^{(r+1)} - S_{n-1}^{(r+1)}$, relation (12) may be written in the form

$$s_n^{(r)} = \alpha_0(n, r)\, S_n^{(r+1)} + [\alpha_1(n, r) - \alpha_0(n, r)] S_{n-1}^{(r+1)} + [\alpha_2(n, r) - \alpha_1(n, r)] S_{n-2}^{(r+1)} + \ldots + [\alpha_{n-1}(n, r) - \alpha_n(n, r)] S_0^{(r+1)}.$$

Moreover, from (4) we have

$$s_n^{(r+1)} = \frac{1}{n+r+1}[s_n^{(r)} + s_{n-1}^{(r)} + \ldots + s_0^{(r)}].$$

Thus

$$\begin{aligned} s_n^{(r+1)} = \frac{1}{n+r+1} \{ & \alpha_0(n, r) S_n^{(r+1)} + [\alpha_1(n, r) - \alpha_0(n, r) + \alpha_0(n-1, r)] S_{n-1}^{(r+1)} \\ & + [\alpha_2(n, r) - \alpha_1(n, r) + \alpha_1(n-1, r) - \alpha_0(n-1, r) + \alpha_0(n-2, r)] S_{n-2}^{(r+1)} \\ & + \ldots \ldots \ldots \ldots \ldots \ldots \\ & + [\alpha_n(n, r) - \alpha_{n-1}(n, r) + \alpha_{n-1}(n-1, r) - \alpha_{n-2}(n-1, r) \\ & + \ldots + \alpha_1(1, r) - \alpha_0(1, r) + \alpha_0(0, r)] S_0^{(r+1)} \}. \end{aligned}$$

The coefficients $\alpha_0(n, r)$, $\alpha_1(n, r)$, ..., $\alpha_n(n, r)$ must therefore satisfy the following system of equations:

$$\begin{aligned} \alpha_m(n, r+1) = \frac{1}{n+r+1} [& \alpha_m(n, r) - \alpha_{m-1}(n, r) + \alpha_{m-1}(n-1, r) \\ & - \alpha_{m-2}(n-1, r) + \ldots + \alpha_0(n-m+1, r) + \alpha_0(n-m, r)]; \end{aligned}$$
$$(m = 0, 1, 2, 3, \ldots, n). \tag{13}$$

Now we know in particular that $s_n^{(1)} = \dfrac{S_n^{(1)}}{n+1}$. Whence, $\alpha_0(n, 1) = \dfrac{1}{n+1}$, $\alpha_1(n, 1) = \alpha_2(n, 1) = \ldots = \alpha_n(n, 1) = 0$, and, starting with these values (corresponding to $r = 1$), we may compute by means of (13) the values of $\alpha_0(n, r)$, $\alpha_1(n, r)$, ..., $\alpha_n(n, r)$.

Thus we obtain

$$\alpha_0(n, r) = \frac{1}{(n+r)(n+r-1)\ldots(n+1)}, \tag{14}$$

and when $r \geqq 2$ $(1 \leqq m \leqq n)$,

$$\alpha_m(n, r) = \frac{1}{(n+r)(n+r-1)\ldots(n-m+1)} \sum_{g=1}^{m} \tau_g(r) P_{m-g}(n, m, r), \tag{15}$$

where $P_0(n, m, r) = 1$ and $P_g(n, m, r)$ $(1 \leqq g \leqq m)$ represents the sum of the products of the factors $(n+r-1)$, $(n+r-2)$, ..., $(n+r-m+1)$ taken g at a time, and where $\tau_1(r)$, $\tau_2(r)$, ..., $\tau_{r-1}(r)$ are functions of r only determined by the following relations:

$$\left.\begin{aligned}
\tau_1(r) &= 1+2+3+\ldots\ldots+(r-1) = \tau_1(r-1)+(r-1),\\
\tau_2(r) &= 2\tau_1(2)+3\tau_1(3)+4\tau_1(4)+\ldots\ldots+(r-1)\tau_1(r-1)\\
&\qquad\qquad = \tau_2(r-1)+(r-1)\tau_1(r-1),\\
\tau_3(r) &= 3\tau_2(3)+4\tau_2(4)+5\tau_2(5)+\ldots\ldots+(r-1)\tau_2(r-1)\\
&\qquad\qquad = \tau_3(r-1)+(r-1)\tau_2(r-1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\tau_{r-1}(r) &= (r-1)\tau_{r-2}(r-1) = (r-1)\,!\\
\tau_r(r) &= \tau_{r+1}(r) = \tau_{r+2}(r) = \ldots\ldots = \tau_n(r) = 0.
\end{aligned}\right\} \quad (16)$$

It may be observed at this point that if we place

$$T_n^{(r)} = \frac{S_n^{(r)}}{(n+r)(n+r-1)\ldots\ldots(n+1)},$$

we shall have, as a result of (12), (14) and (15),

$$s_n^{(r)} = T_n^{(r)} + \beta_1(n, r)T_{n-1}^{(r)} + \ldots\ldots + \beta_n(n, r)T_0^{(r)}, \quad (17)$$

where

$$\beta_m(n, r) = \frac{1}{(n+r)(n+r-1)\ldots\ldots(n+r-m+1)}\sum_{g=1}^{m}\tau_g(r)P_{m-g}(n, m, r);$$
$$(1 \leqq m \leqq n), \quad (18)$$

the expressions $P_g(n, m, r)$ and $\tau_g(r)$ being those indicated above.

In view of these results we may now state the original problem of § 1 as follows, at least when the expressions $s_0^{(r)}$, $s_1^{(r)}$, ..., $s_n^{(r)}$ are defined by (4):

(a) Given

$$S_n^{(r)} = \sum_{m=0}^{n} a_m(n, r)s_{n-m}^{(r)},$$

where $a_m(n, r)$ $(0 \leqq m \leqq n)$ is determined by (9) and (10). To show that if $\lim_{n=\infty} s_n^{(r)} = l$ then $\lim_{n=\infty} S_n^{(r)}/D_n^{(r)} = l$. This last equation may also be written in the form $\lim_{n=\infty} S_n^{(r)}/n^r = l/r\,!$ since

$$\frac{S_n^{(r)}}{D_n^{(r)}} = \frac{r\,!\,S_n^{(r)}}{(n+1)(n+2)\ldots\ldots(n+r)}.$$

(b) Given

$$s_n^{(r)} = \sum_{m=0}^{n}\beta_m(n, r)T_{n-m}^{(r)},$$

where $\beta_0(n, r) = 1$ and $\beta_m(n, r)$ $(1 \leqq m \leqq n)$ is determined by (18). To show that if $\lim\limits_{n=\infty} T_n^{(r)} = l$ then $\lim\limits_{n=\infty} s_n^{(r)} = r!\,l$.

III. *Auxiliary Theorems on Limits.*

3. In order to prove statements (a) and (b) of § 2 we shall make use of the following general theorem in the theory of limits:

(I) Let $c_1, c_2, c_3, \ldots, c_n$ be a sequence of quantities (real or complex) such that $\lim\limits_{n=\infty} c_n = g$, and let $b_1(n), b_2(n), b_3(n), \ldots, b_n(n)$ be a set of n functions of n each of which is positive when n is positive, and such that

$$\lim_{n=\infty} \frac{b_1(n) + b_2(n) + b_3(n) + \ldots + b_m(n)}{b_1(n) + b_2(n) + b_3(n) + \ldots + b_n(n)} = 0\,; \quad m = \textit{constant} \geqq 0.$$

Then

$$\lim_{n=\infty} \frac{b_1(n)c_1 + b_2(n)c_2 + b_3(n)c_3 + \ldots + b_n(n)c_n}{b_1(n) + b_2(n) + b_3(n) + \ldots + b_n(n)} = g.$$

In fact, we shall have under the above hypotheses $c_n = g + \varepsilon_n$, $\lim\limits_{n=\infty} \varepsilon_n = 0$; so that, having chosen an arbitrarily small positive constant ε, we may determine a value m such that for all $n > m$ we may write the following relations, in which for brevity we place $b_1(n) = b_1$, $b_2(n) = b_2$, etc.:

$$\left|\frac{b_1c_1 + b_2c_2 + \ldots + b_nc_n}{b_1 + b_2 + \ldots + b_n} - g\right|$$

$$= \left|\frac{b_1c_1 + b_2c_2 + \ldots + b_mc_m - g(b_1 + b_2 + \ldots + b_m) + \varepsilon_{m+1} b_{m+1} + \varepsilon_{m+2} b_{m+2} + \ldots + \varepsilon_n b_n}{b_1 + b_2 + \ldots + b_n}\right|$$

$$< \left|\frac{b_1c_1 + b_2c_2 + \ldots + b_mc_m - g(b_1 + b_2 + \ldots + b_m)}{b_1 + b_2 + \ldots + b_m}\right| + \varepsilon \frac{b_{m+1} + b_{m+2} + \ldots + b_n}{b_1 + b_2 + \ldots + b_n}$$

$$< (C + g)\frac{b_1 + b_2 + \ldots + b_m}{b_1 + b_2 + \ldots + b_n} + \varepsilon\left(1 - \frac{b_1 + b_2 + \ldots + b_m}{b_1 + b_2 + \ldots + b_n}\right),$$

where C is the greatest of the quantities $|c_1|, |c_2|, \ldots, |c_m|$. Thus the truth of (I) becomes evident.

We add that in case the functions $b_1(n), b_2(n), \ldots, b_n(n)$ are not assumed to be positive, we obtain in like manner the following result:

(II) Let $c_1, c_2, c_3, \ldots, c_n$ be a sequence of quantities (real or complex) such that $\lim\limits_{n=\infty} c_n = g$ and let $b_1(n), b_2(n), \ldots, b_n(n)$ be a set of n functions of n satisfying the following two conditions:

$$\lim_{n=\infty} \frac{|b_1(n)| + |b_2(n)| + \ldots + |b_m(n)|}{b_1(n) + b_2(n) + \ldots + b_n(n)} = 0; \quad m = constant \geqq 0,$$

$$\frac{|b_{m+1}(n)| + |b_{m+2}(n)| + \ldots + |b_n(n)|}{|b_1(n) + b_2(n) + \ldots + b_n(n)|} < K; \quad \begin{cases} K = constant\ independent\ of\ n, \\ \quad n > n_0 = constant. \end{cases}$$

Then

$$\lim_{n=\infty} \frac{b_1(n)c_1 + b_2(n)c_2 + \ldots + b_n(n)c_n}{b_1(n) + b_2(n) + \ldots + b_n(n)} = g.$$

IV. *Proof of Theorem.*

4. We turn now to the proof of statement (a) of § 2.

From (II) it follows that if $\lim_{n=\infty} s_n^{(r)} = l$ we may write

$$\lim_{n=\infty} \frac{\sum_{m=1}^{n} a_m(n, r) s_{n-m}^{(r)}}{\sum_{m=1}^{n} a_m(n, r)} = l, \tag{19}$$

provided we can show that

$$\lim_{n=\infty} \frac{\sum_{m=n-m_1+1}^{n} |a_m(n, r)|}{\sum_{m=1}^{n} a_m(n, r)} = 0; \quad m_1 = constant\ independent\ of\ n, \tag{20}$$

$$\frac{\sum_{m=1}^{n-m_1} |a_m(n, r)|}{\left|\sum_{m=1}^{n} a_m(n, r)\right|} < K; \quad \begin{cases} K = constant\ independent\ of\ n, \\ \quad n > n_0 = constant. \end{cases} \tag{21}$$

From (5) and (9) the relation (19) is seen to be equivalent to the following:

$$\lim_{n=\infty} \frac{1}{n^r} [S_n^{(r)} - (n + r)(n + r - 1) \ldots (n + 1) s_n^{(r)}] = l \lim_{n=\infty} \sum_{m=1}^{n} a_m(n, r).$$

Thus, statement (a) will follow if we can establish (20) and (21) together with the relation

$$1 + \lim_{n=\infty} \frac{1}{n^r} \sum_{m=1}^{n} a_m(n, r) = \frac{1}{r!}. \tag{22}$$

Now, from (10) we see that when $n = \infty$ the numerator of (20) becomes infinite like n^{r-2}, it being the sum of m_1 terms each of which has this same

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property. However, the denominator of (20) becomes infinite like n^r when $n=\infty$. To see this we note that from (10) we have

$$\sum_{m=1}^{n} a_m(n, r) = \sum_{g=1}^{r-1} \frac{(-1)^g \sigma_g(r)}{(g-1)!} \sum_{m=1}^{n} \frac{(m-1)!}{(m-g)!}(n-m+r)(n-m+r-1)\ldots(n-m+g+1)$$

$$= \sum_{g=1}^{r-1} \frac{(-1)^g \sigma_g(r)}{(g-1)!} \sum_{k=1}^{n} \frac{(n-k)!}{(n-k-g+1)!}(k+r-1)(k+r-2)\ldots(k+g). \quad (23)$$

Moreover, we may write

$$\frac{(n-k)!}{(n-k-g+1)!} = (n-k)(n-k-1)\ldots(n-k-g+2)$$

$$= -\sum_{p=1}^{g} \frac{(-1)^p (g-1)!}{(g-p)!\,(p-1)!} n^{g-p}\{(k+g-1)(k+g-2)\ldots(k+g-p+1)+\theta_p(k)\},$$

where $\theta_p(k)$ is of degree $p-1$ in k. The second member of (23) may therefore be put into the form

$$-\sum_{g=1}^{r-1}(-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^p n^{g-p}}{(g-p)!\,(p-1)!}\left\{\sum_{k=1}^{n}(k+r-1)(k+r-2)\ldots(k+g-p+1)+\theta_p(k)\right\}$$

$$= -\sum_{g=1}^{r-1}(-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^p n^{g-p}}{(g-p)!\,(p-1)!}\left\{\frac{(n+r)(n+r-1)\ldots(n+g-p+1)}{r-g+p}+\psi_p(n)\right\},$$

where $\psi_p(n)$ is of degree p in n. Thus, it not only appears that the denominator of (20) becomes infinite like n^r when $n=\infty$, but that the coefficient of n^r in the same expression is

$$\sum_{g=1}^{r-1}(-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^{p+1}}{(g-p)!\,(p-1)!\,(r-g+p)} = \sum_{g=1}^{r-1}(-1)^g \sigma_g(r)\frac{(r-g)!}{r!}.$$

Equation (22) may therefore be replaced by the following:

$$1+\sum_{g=1}^{r-1}(-1)^g \sigma_g(r)\frac{(r-g)!}{r!} = \frac{1}{r!}. \quad (24)$$

From the above conclusions it appears that equation (20) is satisfied; also equation (21), since the same reasoning shows directly that both the numerator and denominator of (21) become infinite like n^r when $n=\infty$. It remains then but to establish (24).

If we multiply the first member of (24) by $\frac{1}{r+1}$, using for this purpose the form

$$\frac{1}{r+1} = 1 - \frac{r}{r+1} = 1 - r\frac{r\,!}{(r+1)!},$$

and subsequently writing

$$\frac{(r-g)\,!}{(r+1)r!} = \frac{1}{(r+1)!}[(r-g+1)\,! - (r-g)\,(r-g)\,!],$$

we obtain

$$1 - \frac{r\,!}{(r+1)\,!}[r+\sigma_1(r)] + \sum_{g=2}^{r-1}\frac{(-1)^g(r-g+1)\,!}{(r+1)\,!}[(r-g+1)\sigma_{g-1}(r)+\sigma_g(r)] + \frac{(-1)^r}{(r+1)\,!},$$

which, by use of (11), becomes

$$1 + \sum_{g=1}^{r}(-1)^g\sigma_g(r+1)\,\frac{(r-g+1)\,!}{(r+1)\,!}.$$

But this is the expression obtained from the first member of (24) by replacing r by $r+1$, and since (24) is at once verifiable for $r=2$, it follows that it is true for $r=r$. Hence, statement (a) becomes established.

We proceed to establish in an analogous manner statement (b). Since the coefficients $\beta_m(n, r)$ $(0 \leqq m \leqq n)$ are all positive, we may evidently employ in this case the result (I) of § 3.

Thus, we know that if $\lim\limits_{n=\infty} T_n^{(r)} = l$ then

$$\lim_{n=\infty}\frac{\sum\limits_{m=1}^{n}\beta_m(n, r)\,T_{n-m}^{(r)}}{\sum\limits_{m=1}^{n}\beta_m(n, r)} = l, \tag{25}$$

provided that

$$\lim_{n=\infty}\frac{\sum\limits_{m=n-m_1+1}^{n}\beta_m(n, r)}{\sum\limits_{m=1}^{n}\beta_m(n, r)} = 0\,; \quad m_1 = \textit{constant independent of } n.$$

But equation (25) is equivalent to

$$\lim_{n=\infty}[s_n^{(r)} - T_n^{(r)}] = -1 + \lim_{n=\infty} s_n^{(r)} = l \lim_{n=\infty}\sum_{m=1}^{n}\beta_m(n, r).$$

The desired result (b) will therefore follow if we can establish (26) together with the relation

$$1 + \lim_{n=\infty} \sum_{m=1}^{n} \beta_m(n, r) = r!. \tag{27}$$

Moreover, relation (26) will follow at once as soon as (27) is established, since it appears from (18) that the numerator of (26) vanishes when $n = \infty$, it being the sum of m_1 terms each of which has this property. It therefore remains but to establish (27).

If we now write the expression for

$$\sum_{m=1}^{n} \beta_m(n, r)$$

as determined by (18) it is readily seen to be of the form

$$\sum_{g=1}^{r-1} f_g(n, r)\, \tau_g(r),$$

where $f_1(n, r) = \dfrac{n}{n+r}$ and $f_g(n, r)$ $(2 \leqq g \leqq r-1)$ is determined by the following:

$$f_g(n, r) = \frac{1}{n+r}\left[f_{g-1}(n-1) + f_{g-1}(n-2) + \dots + f_{g-1}(1)\right].$$

Thus it appears that $\lim_{n=\infty} f_g(n, r) = 1$, so that (27) may be replaced by

$$1 + \sum_{g=1}^{r-1} \tau_g(r) = r!.$$

This relation is now easily established, since we have

$$\begin{aligned}(1+r)\left[1 + \sum_{g=1}^{r-1} \tau_g(r)\right] \equiv 1 + [\tau_1(r) + r] + [\tau_2(r) + r\,\tau_1(r)] + \dots \\ + [\tau_{r-1}(r) + r\,\tau_{r-2}(r)] + r\,\tau_{r-1}(r),\end{aligned}$$

which from (16) reduces to

$$1 + \sum_{g=1}^{r} \tau_g(r+1).$$

The theorem of § 1 thus becomes established, provided the expressions $s_0^{(r)}$, $s_1^{(r)}$, $s_2^{(r)}$, ..., $s_n^{(r)}$ are defined by (4).

5. It remains to show that the systems (3) and (4) are equivalent in the sense in which we have just shown (2) and (4) to be equivalent. In order to avoid duplicity of notation let us suppose hereafter that the expressions $s_n^{(0)}$, $s_n^{(1)}$, $s_n^{(2)}$, ..., $s_n^{(r)}$ of (3) are called $y_n^{(0)}$, $y_n^{(1)}$, $y_n^{(2)}$, ..., $y_n^{(r)}$ respectively.

We proceed to determine n coefficients $\gamma_0(n, r), \gamma_1(n, r), \ldots, \gamma_n(n, r)$, each a function of n and r and such that for all possible values of these quantities we shall have

$$y_n^{(r)} = \gamma_0(n, r)\, s_n^{(r)} + \gamma_1(n, r)\, s_{n-1}^{(r)} + \ldots + \gamma_n(n, r)\, s_0^{(r)}. \tag{28}$$

The process by which this may be carried out closely resembles that by which the coefficients $a_0(n, r), a_1(n, r), \ldots, a_n(n, r)$ of (5) were determined, and we shall therefore merely indicate the essential steps.

From the relations

$$\left.\begin{aligned} s_n^{(r)} &= (n+1)\, s_n^{(r+1)} - n\, s_{n-1}^{(r+1)}, \\ y_n^{(r+1)} &= \frac{1}{n+r+1}\left[y_n^{(r)} + y_{n-1}^{(r)} + \ldots + y_0^{(r)}\right], \end{aligned}\right\} \tag{29}$$

in conjunction with (28), we find that

$$\begin{aligned} \gamma_m(n, r+1) = \frac{n-m+1}{n+r+1}\,&[\gamma_m(n, r) - \gamma_{m-1}(n, r) + \gamma_{m-1}(n-1, r) \\ &- \gamma_{m-2}(n-1, r) + \ldots - \gamma_0(n-m+1, r) + \gamma_0(n-m, r)]\,; \\ &(m = 0, 1, 2, 3, \ldots, n). \end{aligned} \tag{30}$$

Now, we have by inspection $\beta_0(n, 2) = \frac{n+1}{n+2}$. Thus

$$\gamma_0(n, r) = \frac{(n+1)^{r-1}}{(n+2)(n+3)\ldots(n+r)}.$$

Moreover, since we have in like manner $\gamma_1(n, 2) = 0$, while $\gamma_0(n, r)$ as just determined is such that $\gamma_0(n, r) - \gamma_0(n-1, r)$ vanishes like $1/n^2$ when $n = \infty$, it follows from (30) that when $n = \infty$ the coefficient $\gamma_1(n, r)$ vanishes like $1/n^2$. Similarly, we find that $\gamma_2(n, r)$ has the same property. More generally, it appears that when $n = \infty$ the coefficient $\gamma_m(n, r)$ $(1 \leqq m \leqq n)$ vanishes to as high an order as that of the expression

$$\frac{n-m+1}{n+r}\left(\frac{1}{n^2}\right)(1 + 2 + 3 + \ldots + m),$$

the order of which is the same as that of $\frac{m}{n^2}$. In other words,

$$\lim_{n=\infty} \frac{n^2 \gamma_m(n, r)}{m} = \textit{constant } (0 \textit{ incl.})\,; \qquad (1 \leqq m \leqq n).$$

When $\lim_{n=\infty} s_n^{(r)} = l$ we may therefore write

$$\lim_{n=\infty} \sum_{m=1}^{n} \gamma_m(n, r)\, s_{n-m}^{(r)} = 0,$$

and hence

$$\lim_{n=\infty} y_n^{(r)} = \lim_{n=\infty} \beta_0(n, r)\, s_n^{(r)} = \lim_{n=\infty} s_n^{(r)} = l.$$

In order to establish the converse of this result, we proceed to determine the coefficients $\delta_m(n, r)$ for which the following relation is satisfied:

$$s_n^{(r)} = \delta_0(n, r)\, y_n^{(r)} + \delta_1(n, r)\, y_{n-1}^{(r)} + \ldots + \delta_n(n, r) y_0^{(r)}:$$

From the relations

$$y_n^{(r)} = (n + r + 1) y_n^{(r+1)} - (n + r) y_{n-1}^{(r+1)},$$

$$s_n^{(r+1)} = \frac{1}{n+1}\left[s_n^{(r)} + s_{n-1}^{(r)} + \ldots + s_0^{(r)}\right],$$

we find that

$$\delta_m(n, r + 1) = \frac{n + r - m + 1}{n + 1}\left[\delta_m(n, r) - \delta_{m-1}(n, r) + \delta_{m-1}(n - 1, r) - \ldots + \delta_0(n - m, r\right]; \qquad (m = 0, 1, 2, \ldots, n). \quad (31)$$

Equations (31), together with the special equations $\delta_0(n, 2) = \dfrac{n+2}{n+1}$, $\delta_1(n, 2) = \delta_2(n, 2) = \ldots = \delta_n(n, 2) = 0$, enable us to show as in the preceding case that if $\lim_{n=\infty} s_n^{(r)} = l$ then $\lim_{n=\infty} y_n^{(r)} = l$, thus completing the proof of the theorem of § 1.

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